Learning to Love Logarithms

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I don't know where I came up with this title. They make us submit proposals for talks ten months in advance and at that point you think "I can talk about anything."

When I was in junior high school and was doing very well in math and couldn't understand why everyone didn't, my mother, a very intelligent well-educated woman who excelled in fields other than mathematics said to me, "Just wait till you get to logarithms. Then you won't understand it so well." Like just wait till you have children of your own ...

One night I was watching "The West Wing" and two characters were sparring verbally (about football?). One of them had gone to Berkeley and one to Princeton. The Princeton grad said, "My tigers can beat your bears anytime." And the Berkeley grad replied, "At what, logarithms?"

If math is a metaphor for things arcane and obscure and incomprehensible, not to mention boring, then logarithms exemplify that.

I want to talk today about logarithms and how they have been important historically in mathematics and how they are important in mathematics today for an entirely different reason (which I find very intriguing). I believe that, just as we can have more success in helping students learn mathematics if we make it more meaningful and interesting, we can help students learn this somewhat intimidating topic if we put it in a context they can understand.

But first I want to address the issue of loving logarithms. One reason we can love them is because they are beautiful, not just mathematically but esthetically. Consider the logarithmic spiral. This is what it looks like. This is a spiral with the property that any straight line through the origin will intersect a logarithmic spiral at the same angle. The equation of this spiral is

[Picture of logarithmic spiral]

$$\mathbf{r = e^{a\theta}}$$

in polar coordinates. The constant **a** determines the rate of growth of the spiral; the sign of **a** determines if the spiral is right or left-handed.

Logarithmic spirals occur many places in nature: the shell of the chambered nautilus, the arrangement of florets in the core of daisy blossoms, and the arms of spiral galaxies. They also occur in art, such as in Escher's print "Whirlpools". [*See references.]*

Three Important Facts about Logarithms

They have played an important role in the history of mathematics. (And science it turns out)

They are important in mathematics today.

They are not hard if you remember one simple fact.

Let me start at the bottom.

Logarithms are Exponents

$\log_2 8 = 3$ means $2^3 = 8$

What do you do when you see $\text{Log}_3\, x = 4$?

You think $3^4 = x$. Aha!

The Simple Truth: Logarithms are Exponents
And we are not afraid of exponents

Why are logarithms so hard for so many people? I think one reason is the notation. I had one student who would consistently say

log 2 to the 8th for $\log_2 8$

So while logs are exponents the notation doesn't imply the correct relationship between the base and the exponent.

I think another reason is that they are transcendental functions. That is, we can add, subtract, multiply, and divide and still can't get to the bottom of them. Transcendental functions are hard, harder than algebraic, but students must learn to deal with them or they're going to be lost in trigonometry beyond memorizing a few values. I think there are some mathematical leaps we ask students to make, times when they're asked not just to do more of the same, not just to work longer or harder problems, but to make a real cognitive leap. I think negative numbers are one such leap and the idea that a letter can be a number is another. And transcendental functions are yet another.

But if we remember that they're exponents ... they're not so hard.

Logarithms have been important in the history of mathematics

Logarithms are exponents, the power to which a number, such as 10 (called the base), is raised to yield a given number. Since exponents are added when two numbers are multiplied together, the logarithm of a product is the sum of the logarithms of the factors. Likewise, when one number is divided by another, the exponent of the divisor is subtracted from the exponent of the dividend. Calculations using logarithms involve adding and subtracting instead of multiplying and dividing. Logarithms can also be used to raise numbers to powers or extract roots by multiplying and dividing. Using logarithms replaces more complicated computations with simpler ones.

The scientific advances of the sixteenth and seventeenth centuries (Copernicus, Galileo, Kepler) involved greater and greater amounts of numerical data so scientists were spending much of their time doing tedious calculations. We don't know exactly how the Scottish mathematician John Napier (1550 - 1617) first developed logarithms but he knew about

$\sin A \sin B = \_[\cos(A - B) - \cos(A + B)]$ (Identities like this were known in 11th C Egypt)
and that $q^m q^n = q^{m+n}$
for positive integers. Napier extended this rule to real numbers.

Napier spent the years from 1594 to 1614 computing log tables using base $1 - 10^{-7}$. An English mathematician named Henry Briggs (1561 – 1631) convinced Napier to use 10 as a base. In 1624 Briggs published base 10 logarithms from 1 – 20,000 and 90,000 to 100,000. The gap was filled by a Dutch publisher Adriaan Vlacq in 1628 and this was the basis for logarithms until 1949 when a set of tables to 20 places was published in England.

Within a few decades following Briggs' publication scientists and mathematicians throughout the world were using logarithms for their calculations. Kepler was one of the first to use logarithms and his Third Law can be represented logarithmically. Anyone using logarithms for computations had to use the tables to look up the logarithm of each number in the calculation. This could be a tedious task (though not as tedious as calculations without logarithms) and the tables did contain errors. This inspired the eminent French mathematician Pierre Laplace said, "By shortening the labors, the invention of logarithms doubled the life of the astronomer."

But logarithms had yet another contribution to make. An English astronomy and mathematics teacher named Edmund Gunter (1581 – 1626) plotted logarithms of numbers on a line (called Gunter's Line of Numbers) and multiplied and divided numbers by adding and subtracting lengths on the line. An English clergyman named William Oughtred (1574 – 1660) refined Gunter's line by using two pieces of wood which slid against each other. Each piece of wood contained a scale in which the distance of a number from the end of its line is proportional to its logarithm. To multiply two numbers, one of the numbers is lined up with 1 and the product appears opposite the other number. Division reverses the process. The slide rule was not accurate to many decimal places. It also required the user to keep track of where the decimal point belonged.

The slide rule does not count, or add and subtract, like the counting tables and the abacus. Distance on a scale represents a number and the distance is determined by the logarithm of that number. Devices such as this are called analog computers. A clock with hands is an analog computer, with a thirty-degree movement of the minute hand representing five minutes.

Despite its drawbacks, the slide rule was enormously successful. It eliminated the need for using tables of logarithms. The slide rule was used by scientists and mathematicians, as well as students, for over three hundred years until it was replaced by the electronic hand-held calculator beginning in the 1970s.

Logarithms Today

Logarithms may have outlived their usefulness as computational devices but they are far from finished! Since they are the inverse of the enormously useful exponential function they are helpful in solving exponential equations.

One application of exponential equations is carbon 14 dating. The following is the equation for dating a piece of bone in which the carbon 14 has decreased from 100 mg. to 70 mg. The numbers 5570 and 0.5 represent the fact that the half life of bone is 5570 years. A is the current amount of carbon 14 in a substance and A_0 is the original amount.

$$A = A_0 (0.5)^{t/5570}$$

Solving for t given that $A = 70$ and $A_0 = 100$: $70 = 100 (0.5)^{t/5570}$

Dividing both sides by 100: $0.7 = (0.5)^{t/5570}$

It isn't at all clear how to solve for t algebraically, but since the logarithm function is one-to-one we can take the log of both sides:

$$\log 0.7 = \log (0.5)^{t/5570}$$

Then applying the exponent property of logarithms: $\log 0.7 = t/5570 \log (0.5)$

Getting the values for the logs from a calculator and applying simple algebra gives t as approximately equal to 2866 years.

There are many other occasions for solving for variables which appear in exponents in equations, such as finding the amount of time it will take an investment at a certain interest rate to reach a given amount or solving problems related to population growth. The laws of logarithms can help us solve these problems.

There are also logarithmic functions which are interesting. Three of these are the following:

1. Computing the pH of a substance $pH = -\log(H^+)$

where H^+ is the concentration of hydrogen ions in the substance.

2. Measuring the percent of light (P) passing through a substance $\log P = -kd$

where k is a constant related to the opacity of the substance and d is the thickness in meters.

3. Measuring the magnitude of an earthquake

We'll return to earthquakes later. Meanwhile, look at this chart.

The distance from the Earth to the Sun is called one astronomical unit (AU). The distances (d) of the other planets to the Sun can be expressed in AUs. The time the Earth takes to circle the Sun is one year; the times of revolution (T) of the other planets can be expressed in Earth years as well:

Planet	**d (km in millions)**	**d (AU)**	**T (in years)**
Mercury	**57.9**	**0.387**	**0.241**
Venus	**108.16**	**0.723**	**0.615**
Earth	**149.6**	**1.000**	**1.000**
Mars	**227.84**	**1.523**	**1.881**
Jupiter	**778.37**	**5.203**	**11.861**
Saturn	**1427.33**	**9.541**	**29.457**
Uranus	**2870.82**	**19.190**	**84.008**
Neptune	**4500.87**	**30.086**	**164.784**
Pluto	**5910.25**	**39.507**	**284.35**

Notice that the farther a planet is from the Sun the more time it takes to complete one revolution around the Sun; this makes sense because it has farther to go. But exactly what is the relationship between a planet's distance from the Sun and its time of revolution? Whatever it is, it doesn't look straightforward.

Now look at an expanded version of the table.

Planet	d (km in millions)	d (AU)	T (in years)	log d	log T
Mercury	57.9	0.387	0.241	-0.4123	-0.6180
Venus	108.16	0.723	0.615	-0.1409	-0.2111
Earth	149.6	1.000	1.000	0	0
Mars	227.84	1.523	1.881	.1827	.2744
Jupiter	778.37	5.203	11.861	.7163	1.074
Saturn	1427.33	9.541	29.457	.9796	1.469
Uranus	2870.82	19.190	84.008	1.283	1.924
Neptune	4500.87	30.086	164.784	1.478	2.217
Pluto	5910.25	39.507	284.35	1.597	2.454

If you look at the last two columns, log d and log T, you will notice that Log T is approximately 1.5 times log d. Actually, it is exact, but the values for d, T, log d, and log T are approximate. Let's look at this relationship a bit further.

$\log T = 1.5 \log d$
$2 \log T = 3 \log d$
Applying the exponent rule for logarithms: $\log T^2 = \log d^3$
Since the logarithm function is one-to-one: $T^2 = d^3$

You have just discovered Kepler's Third Law of planetary motion. This gives us the relationship between a planet's distance from the Sun and the time it takes the planet to orbit the Sun. While this relationship doesn't require logarithms, you can see that using logs makes it easy to discover the rule. Note that Kepler's Third Law is still valid if we measure time and distance in units other that Aus and Earth years, but we will need a constant.

Earthquakes

Earthquakes are measured on something called the Richter scale. [See the references for a nomogram which illustrates the magnitude of earthquakes.] I live in Earthquake Country and most people there know this scale isn't linear. You could sleep through a magnitude 4 quake while an 8.0 would be devastating. The formula for computing the magnitude of an earthquake is

$$M = \log_{10} A(mm) + (\textit{Distance correction factor})$$

where A(mm) is the amplitude in millimeters that the arm of a seismograph moves. An example of this formula applied to a location in Southern California is

$$M = \log_{10} A(mm) + 3 \log_{10} [8\Delta t(s)] - 2.92$$

Here is some data from some recent US earthquakes.

Quake	**Magnitude**	**Depth (miles)**	**Deaths**	**Damage (Billion $)**
Loma Prieta CA 1989	**6.9**	**10.5**	**63**	**6**
Northridge CA 1994	**6.7**	**11**	**57**	**40**
Nisqually (Seattle) WA 2001	**6.8**	**33**	**0**	**2**

There doesn't seem to be much relationship between the magnitude of the quake and either the damage in dollars or the number of deaths. Here are the facts behind these numbers: The Northridge quake was in a heavily populated area and so had a large amount of damage, but it occurred at 4:30 am so there were fewer casualties even though numerous structures collapsed. The Loma Prieta quake happened about 5 pm on a weekday but was centered in an unpopulated area in the Santa Cruz Mountains. If the epicenter had been closer to downtown San Francisco, the cost in both dollars and human lives could have been devastating. The most significant fact about the Nisqually quake was the depth. Seattle was very lucky.

I was in both the Loma Prieta and Nisqually quakes, about 35 miles from the epicenter of each. The Loma Prieta quake felt FAR more threatening. I spent the Loma Prieta quake under a table hearing the ceiling of the room I was in fall in chunks onto the tabletop. The Nisqually quake felt like the type we in San Francisco barely notice.

In a 1999 article in the San Francisco *Chronicle* entitled "Time to Dump the Richter Scale" the author argued that the Richter scale is misleading because it makes a 6.9 (Loma Prieta 1989) quake and a 7.6 (Taipei 1999) one seem to be of nearly equal magnitude when actually the second is eight times more powerful than the first. The velocity of a 20 mph wind is twice the velocity of a 10 mph wind. Why can't we have the same relationship with earthquakes?
Then, the new scale would mean that a Richter magnitude 1 would be a 1 on the new scale. A Richter 2 would be a 30; a 3 would be a 900; 4 would be 27,000. The largest recorded earthquake, a 9.5 on the Richter scale, would be 3.6 trillion.

The author then points out that if other areas of science used scales similar to the Richter, the age of the universe would no longer be 15,000,000,000 years but 10.17 Carl Sagans. And our computer would not have 20 giga bytes of memory but 10.3 Bill Gates.

I hope you have learned something about logarithms. The next page contains some references you may find interesting.

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See http://www.seismo.unr.edu/ftp/pub/louie/class/100/magnitude.html *This web site contains information on earthquakes and includes a nomogram.*

http://www.sfgate.com/cgi-bin/article.cgi?file=/chronicle/archive/1999/10/17/ED71817.DTL *This is the URL of the article "Time to Dump the Richter Scale."*